Convex sets with homothetic projections

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Abstract. Nonempty sets X_1 and X_2 in the Euclidean space $\mathbb{R}^n$ are called *homothetic* provided $X_1 = z + \lambda X_2$ for a suitable point $z \in \mathbb{R}^n$ and a scalar $\lambda \neq 0$, not necessarily positive. Extending results of Süss and Hadwiger (proved by them for the case of convex bodies and positive λ), we show that compact (respectively, closed) convex sets K_1 and K_2 in $\mathbb{R}^n$ are homothetic provided for any given integer m, $2 \le m \le n-1$ (respectively, $3 \le m \le n-1$), the orthogonal projections of K_1 and K_2 on every m-dimensional plane of $\mathbb{R}^n$ are homothetic, where homothety ratio may depend on the projection plane. The proof uses a refined version of Straszewicz's theorem on exposed points of compact convex sets.

AMS Subject Classification: 52A20.

Keywords: Convex set, exposed points, homothety, projection.

1 Introduction and main results

Let us recall that nonempty sets X_1 and X_2 in the Euclidean space $\mathbb{R}^n$ are *homothetic* provided $X_1 = z + \lambda X_2$ for a suitable point $z \in \mathbb{R}^n$ and a scalar $\lambda \neq 0$ (called homothety ratio); furthermore, X_1 and X_2 are called *positively* homothetic (respectively, *negatively* homothetic) provided $\lambda > 0$ (respectively, $\lambda < 0$). We remark that in convex geometry homothety usually means positive homothety, also called direct homothety. In a standard way, a convex body in $\mathbb{R}^n$ is a compact convex set with nonempty interior.

Süss [15, 16] proved that a pair of convex bodies in $\mathbb{R}^n$, $n \ge 3$, are positively homothetic if and only if the orthogonal projections of these bodies on every hyperplane are positively homothetic, where homothety ratio may depend on the projection hyperplane (the proof of this statement is given for $n = 3$ with the remark in [16, p. 49] that the extension to higher dimensions is routine). Following a series of intermediate results (see [13] for additional references), Hadwiger [4] showed that convex bodies K_1 and K_2 in $\mathbb{R}^n$ are positively homothetic if and only if there is an integer m, $2 \le m \le n-1$, such that the orthogonal projections of K_1 and K_2 on each m-dimensional plane are positively homothetic (see also Rogers [10] for the case $m = 2$).

The question whether the statements of Süss and Hadwiger hold for larger families of geometric transformations in $\mathbb{R}^n$, like similarities, was posed by Nakajima [7, p. 169] for $n = 3$ and independently by Petty and McKinney [9] and

Golubyatnikov [2]. Gardner and Volčič [1] showed the existence of a pair of centered and coaxial convex bodies of revolution in $\mathbb{R}^n$ whose orthogonal projections on every 2-dimensional plane are similar, but which are not themselves even affinely equivalent. On the other hand, Golubyatnikov [2, 3] proved that compact convex sets K_1 and K_2 in $\mathbb{R}^n$ are homothetic (positively or negatively) provided their projections on every 2-dimensional plane are similar and have no rotation symmetries.

Our first theorem shows that the family of positive homotheties in Süss's and Hadwiger's statements can be extended to all homotheties in $\mathbb{R}^n$.

Theorem 1. *Given compact* (*respectively, closed*) *convex sets* K_1 *and* K_2 *in* $\mathbb{R}^n$ *and an integer* m, $2 \le m \le n-1$ (*respectively,* $3 \le m \le n-1$), *the following conditions are equivalent:*

1) K_1 *and* K_2 *are homothetic,*

2) *the orthogonal projections of* K_1 *and* K_2 *on every* m*-dimensional plane of* $\mathbb{R}^n$ *are homothetic, where homothety ratio may depend on the projection plane.*

The following example shows that the inequality $m \ge 3$ in Theorem 1 is sharp for the case of unbounded convex sets.

Example 1. Let K_1 and K_2 be solid paraboloids in $\mathbb{R}^3$, given, respectively, by

$$K_1 = \{(x, y, z) \mid x^2 + y^2 \le z\} \quad \text{and} \quad K_2 = \{(x, y, z) \mid 2x^2 + y^2 \le z\}.$$

Obviously, K_1 and K_2 are not homothetic. At the same time, their orthogonal projections $\pi_L(K_1)$ and $\pi_L(K_2)$ on every 2-dimensional plane $L \subset \mathbb{R}^3$ are positively homothetic. Indeed, if $L = \{(x, y, z) \mid z = \text{const}\}$, then $\pi_L(K_1) = \pi_L(K_2) = L$. For any other 2-dimensional plane L in $\mathbb{R}^3$, the sets $\pi_L(K_1)$ and $\pi_L(K_2)$ are closed convex sets bounded by parabolas whose axes of symmetry are parallel to the orthogonal projection of the z-axis on L. Since any two parabolas in the plane with parallel axes of symmetry are homothetic, the sets $\pi_L(K_1)$ and $\pi_L(K_2)$ also are positively homothetic.

In view of this example, it would be interesting to describe the pairs of closed convex sets K_1 and K_2 in $\mathbb{R}^n$ such that the orthogonal projections of K_1 and K_2 on every 2-dimensional plane of $\mathbb{R}^n$ are homothetic. The following corollary slightly refines Theorem 1.

Corollary 1. *Given compact* (*respectively, closed*) *convex sets* K_1 *and* K_2 *in* $\mathbb{R}^n$, *integers* r *and* m *such that* $0 \le r \le m-2 \le n-3$ (*respectively,* $0 \le r \le m-3 \le n-4$), *and a subspace* $S \subset \mathbb{R}^n$ *of dimension* r, *the following conditions are equivalent:*

1) K_1 *and* K_2 *are homothetic,*

2) *the orthogonal projections of* K_1 *and* K_2 *on every* m*-dimensional plane of* $\mathbb{R}^n$ *that contains* S *are homothetic, where homothety ratio may depend on the projection plane.*

We observe that the proof of Theorem 1 cannot be routinely reduced to that of Süss and Hadwiger by using compactness and continuity arguments. Indeed, if orthogonal projections $\pi_L(K_1)$ and $\pi_L(K_2)$ of the convex sets K_1 and K_2 on a plane $L \subset \mathbb{R}^n$ are homothetic and

$$\pi_L(K_1) = z_L + \lambda_L \pi_L(K_2),$$

then z_L and λ_L (but not the absolute value of λ_L) may loose their continuity as functions of L when both $\pi_L(K_1)$ and $\pi_L(K_2)$ become centrally symmetric. To avoid the consideration of centrally symmetric projections, our proof of Theorem 1 uses a refined version of Straszewicz's theorem on exposed points of a compact convex set (see Theorem 2 below).

Let us recall that a point x of a closed convex set $K \subset \mathbb{R}^n$ is called *exposed* provided there is a hyperplane $H \subset \mathbb{R}^n$ supporting K such that $H \cap K = \{x\}$. Straszewicz's theorem states that any compact convex set in $\mathbb{R}^n$ is the closed convex hull of its exposed points (see [17]). Klee [5] proved that a line-free closed convex set $K \subset \mathbb{R}^n$ is the closed convex hull of its exposed points and exposed halflines (a set is called *line-free* if it contains no lines).

Points x and z of a compact convex set $K \subset \mathbb{R}^n$ are called (affinely) *antipodal* provided there are distinct parallel hyperplanes H and G both supporting K such that $x \in H \cap K$ and $z \in G \cap K$ (see, e. g., [6, 12] for various antipodality properties of convex and finite sets in $\mathbb{R}^n$). Furthermore, the points x and z are called *antipodally exposed* (and the chord $[x, z]$ is called an *exposed diameter* of K) provided the parallel hyperplanes H and G can be chosen such that $H \cap K = \{x\}$ and $G \cap K = \{z\}$ (see [8, 11]). Clearly, a compact convex set may have exposed points which are not antipodally exposed (like the point x in Figure 1).

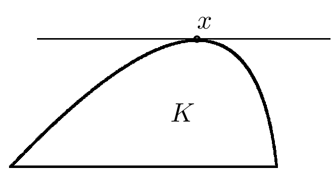

Figure 1: An exposed point which is not antipodally exposed.

Theorem 2. *Any compact convex set $K \subset \mathbb{R}^n$ distinct from a singleton is the closed convex hull of its antipodally exposed points.*

In what follows, we will need the following lemma.

Lemma 1. *No two distinct exposed diameters of a compact convex set $K \subset \mathbb{R}^n$ are parallel.*

Proof. Assume for a moment that K has a pair of distinct parallel exposed diameters, say $[x_1, z_1]$ and $[x_2, z_2]$. We may suppose that $x_1 - z_1$ and $x_2 - z_2$ have the same direction and $\|x_1 - z_1\| \leq \|x_2 - z_2\|$. Denote by H and G distinct parallel hyperplanes both supporting K such that $H \cap K = \{x_1\}$ and

$G \cap K = \{z_1\}$. Let $[x_2', z_2']$ be the intersection of the line (x_2, z_2) and the closed slab between G and H. Clearly, $\|x_2' - z_2'\| = \|x_1 - z_1\|$. Since $[x_2, z_2] \subset K$, we conclude that $[x_2, z_2] \subset [x_2', z_2']$. Then $[x_2, z_2] = [x_2', z_2']$ because of $\|x_1 - z_1\| \le \|x_2' - z_2'\|$. Hence $x_2 \in H$ and $z_2 \in G$. Due to $H \cap K = \{x_1\}$ and $G \cap K = \{z_1\}$, we obtain $[x_1, z_1] = [x_2, z_2]$, in contradiction with the choice of these diameters. $\square$

We conclude this section with necessary definitions, notions, and statements (see, e. g., [18] for general references). In a standard way, $\operatorname{bd} K$, $\operatorname{cl} K$, and $\operatorname{int} K$ denote the boundary, the closure, and the interior of a convex set $K \subset \mathbb{R}^n$; the *recession cone* of K is defined by

$$\operatorname{rec} K = \{y \in \mathbb{R}^n \mid x + \alpha y \in K \text{ for all } x \in K \text{ and } \alpha \ge 0\}.$$

It is well-known that $\operatorname{rec} K \ne \{o\}$ if and only if K is unbounded. The *linearity spaces* $\operatorname{lin} K$ of K is given by $\operatorname{lin} K = (\operatorname{rec} K) \cap (-\operatorname{rec} K)$, and K can be expressed as the direct sum $K = \operatorname{lin} K \oplus (K \cap M)$, where the subspace M is the orthogonal complement of $\operatorname{lin} K$ and $K \cap M$ is a line-free closed convex set

We say that a closed halfspace P of $\mathbb{R}^n$ supports K provided the boundary hyperplane of P supports K and the interior of P is disjoint from K. If the halfspace P is given by $P = \{x \in \mathbb{R}^n \mid x \cdot f \ge \alpha\}$ where f is a unit vector and α is a scalar, then f is called the *inward unit normal* of P. Closed halfspaces S and T in $\mathbb{R}^n$ are called *opposite* provided they can be written as

$$S = \{x \in \mathbb{R}^n \mid x \cdot g \ge \alpha\} \quad \text{and} \quad T = \{x \in \mathbb{R}^n \mid x \cdot g \le \beta\} \tag{1}$$

for a suitable unit vector $g \in \mathbb{R}^n$ and scalars $\alpha \ge \beta$. Clearly, the boundary hyperplanes of opposite halfspaces are parallel. A *plane* in $\mathbb{R}^n$ is a set of the form $F = z + L$, where $z \in \mathbb{R}^n$ and L is a subspace of $\mathbb{R}^n$. For any plane $L \subset \mathbb{R}^n$, we denote by $\pi_L(X)$ the orthogonal projection of a set $X \subset \mathbb{R}^n$ on L. To distinguish similarly looking elements, we write 0 for the real number zero, and o for the origin of $\mathbb{R}^n$.

2 Proof of Theorem 2

We precede the proof of Theorem 2 by two necessary lemmas. One might observe that an alternative proof of Lemma 2 can use a duality argument and the fact the set of regular points of a convex body $K \subset \mathbb{R}^n$ is dense in $\operatorname{bd} K$ (see also [14]).

Lemma 2. *Let $K \subset \mathbb{R}^n$ be a compact convex set and f be a unit vector in $\mathbb{R}^n$. For any $\varepsilon > 0$, there is a closed halfspace $P \subset \mathbb{R}^n$ such that $K \cap P$ is a singleton and the inward unit normal g of P satisfies the inequality $\|f - g\| \le \varepsilon$.*

Proof. Let $Q \subset \mathbb{R}^n$ be the closed halfspace with inward unit normal f that supports K. Denote by H the boundary hyperplane of Q. Choose a point $v \in H \cap K$, and let $U \subset H$ be an $(n-1)$-dimensional closed ball with center v and radius $\delta > 0$ such that the orthogonal projection of K on H lies in U.

Let l be the line through v in the direction of f. Then K entirely lies in the both-way infinite cylinder C with base U and axis l. Choose a closed ball $B_\rho(c)$ with center $c \in l \setminus Q$ and radius $\rho > 0$ such that $K \cup U \subset B_\rho(c)$. Furthermore, we assume that $\rho \geq \delta \cdot \sec \gamma$ where $\gamma = 2 \arcsin(\varepsilon/2)$. If y is a boundary point of $B_\rho(c)$ that lies in $C \cap Q$ and $e_y \in \mathbb{R}^n$ is the unit vector such that $y + e_y$ is the outward unit normal of $B_\rho(c)$ at y, then the inequality $\rho \geq \delta \cdot \sec \gamma$ easily implies that $\|e_y - f\| \leq \varepsilon$.

By continuity, there is a scalar $\alpha \geq 0$ such that the ball $B = B_\rho(c) - \alpha f$ contains K and the boundary of B has at least one, say x, common point with K. Clearly, $x \in C$. Denote by P the closed halfspace of $\mathbb{R}^n$ such that $B \cap P = \{x\}$. By the above, the inward unit normal g of P satisfies the inequality $\|f - g\| \leq \varepsilon$. Finally, $K \cap P = B \cap P = \{x\}$ implies that $K \cap P$ is a singleton (that is, x is an exposed point of K). $\square$

Lemma 3. *Let $K \subset \mathbb{R}^n$ be a compact convex set with more than one point and f be a unit vector in $\mathbb{R}^n$. For any $\varepsilon > 0$, there is a unit vector $g \in \mathbb{R}^n$ and opposite closed halfspaces P and Q both orthogonal to g and supporting K such that $\|f - g\| \leq \varepsilon$ and the sets $K \cap S$ and $K \cap T$ are distinct singletons.*

Proof. Consider the compact convex set $K^* = K + (-K)$. By Lemma 2, there is a closed halfspace $P \subset \mathbb{R}^n$ such that $K^* \cap P$ is a singleton and the inward unit normal g of P satisfies the inequality $\|f - g\| \leq \varepsilon$. Furthermore, $K^* \cap P \neq K^*$ since K has more than one point. Denote by S and $-T$ the closed halfspaces that are translates of P and support the sets K and $-K$, respectively. From

$$K^* \cap P = (K + (-K)) \cap P = K \cap S + (-K) \cap (-T)$$

we conclude that both sets $K \cap S$ and $(-K) \cap (-T)$ are singletons. Finally, $K \cap S$ and $K \cap T$ are distinct due to $K^* \cap P \neq K^*$. $\square$

We start the proof of Theorem 2 by considering the set E of antipodally exposed points of K. Obviously, $\mathrm{cl}\,(\mathrm{conv}\, E) \subset K$; so it remains to show the opposite inclusion. Assume, for contradiction, the existence of a point $a \in K \setminus \mathrm{cl}\,(\mathrm{conv}\, E)$. By the separation properties of convex sets, there is a closed halfspace $Q \subset \mathbb{R}^n$ that contains a and is disjoint from $\mathrm{cl}\,(\mathrm{conv}\, E)$. Denote by Q' the translate of Q that supports K. Clearly, $Q' \subset Q$; so $Q' \cap \mathrm{cl}\,(\mathrm{conv}\, E) = \varnothing$. We can write $Q' = \{x \in \mathbb{R}^n \mid x \cdot f \geq \gamma\}$, where f is the inward unit normal of Q' and γ is a suitable scalar. Since the set $\mathrm{cl}\,(\mathrm{conv}\, E)$ is compact, there is an $\varepsilon > 0$ such that any closed halfspace $P \subset \mathbb{R}^n$ with inward unit normal e is disjoint from $\mathrm{cl}\,(\mathrm{conv}\, E)$ provided P supports K and $\|f - e\| \leq \varepsilon$. By Lemma 3, there is a unit vector g with $\|f - g\| \leq \varepsilon$ and a pair of opposite closed halfspaces S and T of the form (1) such that $K \cap S$ and $K \cap T$ are distinct singletons. If $K \cap S = \{u\}$ and $K \cap T = \{v\}$, then u and v are antipodally exposed points of K. Finally, $S \cap (\mathrm{cl}\, \mathrm{conv}\, E) = \varnothing$ implies $u \notin \mathrm{cl}\,(\mathrm{conv}\, E)$, a contradiction. $\square$

3 Proof of Theorem 1

Obviously, 1) $\Rightarrow$ 2). We start the proof of the converse statement by considering the case when both K_1 and K_2 are compact.

Case I. Both K_1 and K_2 are compact and $2 \le m \le n-1$.

Since 2) trivially implies 1) when both K_1 and K_2 are singletons, we may assume, in what follows, that each of K_1 and K_2 has more than one point.

A) We consider the case $m = n-1$ separately, dividing our consideration into a sequence of steps.

1. First, we state that for any exposed diameter $[x_1, z_1]$ of K_1 and opposite closed halfspaces P_1 and Q_1 of $\mathbb{R}^n$ with the property

$$K_1 \cap P_1 = \{x_1\} \quad \text{and} \quad K_1 \cap Q_1 = \{z_1\},$$

there is an exposed diameter $[x_2, z_2]$ of K_2 parallel to $[x_1, z_1]$ and opposite closed halfspaces P_2 and Q_2 of $\mathbb{R}^n$ that are translates of P_1 and Q_1, respectively, such that

$$K_2 \cap P_2 = \{x_2\} \quad \text{and} \quad K_2 \cap Q_2 = \{z_2\}.$$

Indeed, denote by P_2 and Q_2 some translates of P_1 and Q_1, respectively, that support K_2. Clearly, $P_2 \cap Q_2 = \varnothing$. Choose any points $x_2 \in K_2 \cap P_2$ and $z_2 \in K_2 \cap Q_2$. Assume for a moment that $[x_2, z_2]$ is not parallel to $[x_1, z_1]$. Then the line through x_1 parallel to $[x_2, z_2]$ intersects the hyperplane $\mathrm{bd}\, Q_1$ at a point z_1' distinct from z_1. Choose in $\mathrm{bd}\, Q_1$ a line l through z_1 orthogonal to the line (z_1, z_1') and denote by L the hyperplane through z_1 orthogonal to l. Clearly, the parallel $(n-2)$-dimensional planes $L \cap \mathrm{bd}\, P_i$ and $L \cap \mathrm{bd}\, Q_i$ are distinct and support the orthogonal projection $\pi_L(K_i)$, $i = 1, 2$, such that

$$\begin{gathered}(L \cap \mathrm{bd}\, P_1) \cap \pi_L(K_1) = \{\pi_L(x_1)\}, \quad (L \cap \mathrm{bd}\, Q_i) \cap \pi_L(K_1) = \{\pi_L(z_1)\}, \\ \pi_L(x_2) \in (L \cap \mathrm{bd}\, P_2) \cap \pi_L(K_2), \quad \pi_L(z_2) \in (L \cap \mathrm{bd}\, Q_2) \cap \pi_L(K_2).\end{gathered}$$

By the hypothesis, $\pi_L(K_1)$ and $\pi_L(K_2)$ are homothetic. Hence there is an exposed diameter $[u, v]$ of $\pi_L(K_2)$ parallel to $[\pi_L(x_1), \pi_L(z_1)]$ such that

$$(L \cap \mathrm{bd}\, P_2) \cap \pi_L(K_2) = \{u\}, \quad (L \cap \mathrm{bd}\, Q_2) \cap \pi_L(K_2) = \{v\}.$$

This gives $\pi_L(x_2) = u$ and $\pi_L(z_2) = v$, which is impossible because the line segments $[\pi_L(x_1), \pi_L(z_1)]$ and $[\pi_L(x_2), \pi_L(z_2)]$ are not parallel. The obtained contradiction shows that $[x_2, z_2]$ is parallel to $[x_1, z_1]$ for any choice of $x_2 \in K_2 \cap P_2$ and $z_2 \in K_2 \cap Q_2$. Hence both sets $K_2 \cap P_2$ and $K_2 \cap Q_2$ are singletons, which implies that $[x_2, z_2]$ is an exposed diameter of K_2 parallel to $[x_1, z_1]$.

2. Choose an exposed diameter $[x_0, z_0]$ of K_1 and denote by $[x_0', z_0']$ the exposed diameter of K_2 parallel to $[x_0, z_0]$ (the uniqueness of $[x_0', z_0']$ follows from Lemma 1). Replacing K_1 with $K_1 - (x_0 + z_0)/2$ and K_2 with

$$\lambda(K_2 - (x_0' + z_0')/2), \quad \lambda = \|x_0 - z_0\| / \|x_0' - z_0'\|,$$

we may assume that $[x_0, z_0]$ is an exposed diameter for both K_1 and K_2, centered at o. By **1** above, both K_1 and K_2 are supported by opposite closed halfspaces P_0 and Q_0 such that

$$K_1 \cap P_0 = K_2 \cap P_0 = \{x_0\}, \quad K_1 \cap Q_0 = K_2 \cap Q_0 = \{z_0\}.$$

Applying, if necessary, a suitable affine transformation, we may assume that both hyperplanes $\operatorname{bd} P_0$ and $\operatorname{bd} Q_0$ are orthogonal to $[x_0, z_0]$. Clearly, the orthogonal projections of the transformed sets K_1 and K_2 on any plane are homothetic.

3. We state that any exposed diameter $[x_2, z_2]$ of K_2 is a translate of a suitable exposed diameter $[x_1, z_1]$ of K_1.

Since this statement trivially holds when $[x_2, z_2] = [x_0, z_0]$, we assume, in what follows, that $[x_2, z_2] \neq [x_0, z_0]$. Let P_2 and Q_2 be opposite closed halfspaces of $\mathbb{R}^n$ with the property $K_2 \cap P_2 = \{x_2\}$ and $K_2 \cap Q_2 = \{z_2\}$. Denote by P_1 and Q_1 translates of P_2 and Q_2, respectively, that support K_1. By **1** above, the sets $K_1 \cap P_1$ and $K_1 \cap Q_1$ are singletons, say, $\{x_1\}$ and $\{z_1\}$, such that $[x_1, z_1]$ and $[x_2, z_2]$ are parallel. Clearly, $P_1 \neq P_0 \neq P_2$ and $Q_1 \neq Q_0 \neq Q_2$ due to $[x_2, z_2] \neq [x_0, z_0]$.

Choose a line $l \subset \operatorname{bd} P_0 \cap \operatorname{bd} P_1$ and denote by L the hyperplane through $[x_0, z_0]$ orthogonal to l. Clearly, $\pi_L(K_i)$, $i = 1, 2$, is a compact convex set distinct from a singleton and bounded by two pairs of parallel $(n-2)$-dimensional planes

$$L \cap \operatorname{bd} P_0,\ L \cap \operatorname{bd} Q_0 \quad \text{and} \quad L \cap \operatorname{bd} P_i,\ L \cap \operatorname{bd} Q_i.$$

This shows that both $[\pi_L(x_0), \pi_L(z_0)]$ and $[\pi_L(x_i), \pi_L(z_i)]$ are exposed diameters of $\pi_L(K_i)$, $i = 1, 2$. Since $\pi_L(K_1)$ and $\pi_L(K_2)$ are homothetic and share an exposed diameter $[\pi_L(x_0), \pi_L(z_0)]$, the set $\pi_L(K_2)$ equals one of the sets $\pi_L(K_1)$, $\pi_L(-K_1)$. In either case, $[\pi_L(x_2), \pi_L(z_2)]$ is a translate of $[\pi_L(x_1), \pi_L(z_1)]$. Because $[x_1, z_1]$ and $[x_2, z_2]$ are parallel, we conclude that $[x_2, z_2]$ is a translate of $[x_1, z_1]$.

4. Our next statement (in continuation of **3** above) is that the exposed diameter $[x_2, z_2]$ of K_2 coincides with $[x_1, z_1]$ or with $[-x_1, -z_1]$.

Indeed, by the proved in **3** above, $\pi_L(K_2)$ equals one of the sets $\pi_L(K_1)$, $\pi_L(-K_1)$; whence its exposed diameter $[\pi_L(x_2), \pi_L(z_2)]$ coincides with one of the line segments $[\pi_L(x_1), \pi_L(z_1)]$, $[\pi_L(-x_1), \pi_L(-z_1)]$. Without loss of generality, we may assume that

$$[\pi_L(x_2), \pi_L(z_2)] = [\pi_L(x_1), \pi_L(z_1)]. \tag{2}$$

Let M be the hyperplane through $[x_0, z_0]$ parallel to the $(n-2)$-dimensional plane $\operatorname{bd} P_0 \cap \operatorname{bd} P_1$. Denote by M' a hyperplane (distinct from both L and M) that contains the $(n-2)$-dimensional plane $L \cap M$, and let P_i' and Q_i' be the opposite closed halfspaces of $\mathbb{R}^n$ both supporting K_i whose boundary hyperplanes $\operatorname{bd} P_i'$ and $\operatorname{bd} Q_i'$ are parallel to M', $i = 1, 2$. Consider the hyperplane

L' through $L \cap M$ that forms an angle of $90°$ with M'. If π'_L is the orthogonal projection of $\mathbb{R}^n$ onto L', then the homothetic set $\pi'_L(K_1)$ and $\pi'_L(K_2)$ have $[x_0, z_0]$ as a common exposed diameter, which implies that $\pi'_L(K_2) = \pi'_L(K_1)$ or $\pi'_L(K_2) = \pi'_L(-K_1)$. Clearly, the equality $\pi'_L(K_2) = \pi'_L(K_1)$ gives $P'_2 = P'_1$, and the equality $\pi'_L(K_2) = \pi'_L(-K_1)$ gives $P'_2 = -Q'_1$.

Assume, for contradiction, that

$$[x_1, z_1] \neq [x_2, z_2] \neq [-x_1, -z_1].$$

Due to (2), both lines (x_1, x_2) and (z_1, z_2) are parallel to l. Because x_2 and z_2 are the only points of contact of K_2 with P_2 and Q_2, respectively, there is an $\varepsilon_1 > 0$ so small that if the angle γ between M and M' is positive and less than ε_1, then either $x_1 \in \operatorname{int} P'_2$ or $z_1 \in \operatorname{int} Q'_2$. In either case, $P'_1 \neq P'_2$ for all $\gamma \in]0, \varepsilon_1[$; whence $\pi'_L(K_2) \neq \pi'_L(K_1)$ for all $\gamma \in]0, \varepsilon_1[$.

Under assumption (2), we consider two more subcases.

4a. $[\pi_L(x_2), \pi_L(z_2)] = [\pi_L(-z_1), \pi_L(-x_1)]$ (see part (*i*) of Figure 2).

Then $-z_1 \in (x_1, x_2)$ and $-x_1 \in (z_1, z_2)$. As above, there is a scalar $\varepsilon_2 > 0$ so small that if the angle γ between M and M' is positive and less than ε_2, then either $-z_1 \in \operatorname{int} P'_2$ or $-x_1 \in \operatorname{int} Q'_2$. In either case, $P'_2 \neq -Q'_1$ for all $\gamma \in]0, \varepsilon_2[$; whence $\pi'_L(K_2) \neq \pi'_L(-K_1)$ for all $\gamma \in]0, \varepsilon_2[$.

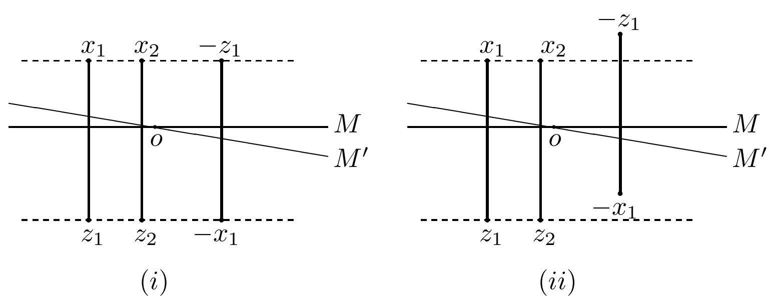

Figure 2: Illustration of subcases 4a and 4b.

4b. $[\pi_L(x_2), \pi_L(z_2)] \neq [\pi_L(-z_1), \pi_L(-x_1)]$ (see part (*ii*) of Figure 2).

In particular, $\pi_L(x_2) \neq \pi_L(-z_1)$. Because of $K_2 \cap P_2 = \{x_2\}$ and $(-K_1) \cap (-Q_1) = \{-z_1\}$, there is a scalar $\varepsilon_3 > 0$ such that if the angle γ between M and M' is positive and less than ε_3, then the compact sets $K_2 \cap P'_2$ and $(-K_1) \cap (-Q'_1)$ are small enough: that is, for any points $u \in K_2 \cap P'_2$ and $v \in (-K_1) \cap (-Q'_1)$,

$$\begin{aligned} \|u - x_2\| &\leq \tfrac{1}{4}\|\pi_L(x_2) - \pi_L(-z_1)\|, \\ \|v - (-z_1)\| &\leq \tfrac{1}{4}\|\pi_L(x_2) - \pi_L(-z_1)\|. \end{aligned} \tag{3}$$

By continuity, ε_3 can be chosen so small that

$$\|\pi'_L(x_2) - \pi'_L(-z_1)\| \geq \tfrac{3}{4}\|\pi_L(x_2) - \pi_L(-z_1)\| \tag{4}$$

for all $\gamma \in]0, \varepsilon_3[$. Together with

$$\|\pi'_L(u) - \pi'_L(x_2)\| \le \|u - x_2\|, \quad \|\pi'_L(v) - \pi'_L(-z_1)\| \le \|v - (-z_1)\|,$$

the inequalities (3) and (4) give

$$\begin{aligned} &\|\pi'_L(u) - \pi'_L(v)\| \\ &\ge \|\pi'_L(x_2) - \pi'_L(-z_1)\| - \|\pi'_L(u) - \pi'_L(x_2)\| - \|\pi'_L(v) - \pi'_L(-z_1)\| \\ &\ge \tfrac{3}{4}\|\pi_L(x_2) - \pi_L(-z_1)\| - \|u - x_2\| - \|v - (-z_1)\| \\ &\ge \tfrac{1}{4}\|\pi_L(x_2) - \pi_L(-z_1)\|. \end{aligned} \tag{5}$$

Since $\pi'_L(K_2)$ is supported by P'_2 and $\pi'_L(-K_1)$ is supported by $-Q'_1$, which is a translate of P'_2, the inequality (5) shows that the contact sets

$$\pi'_L(K_2) \cap P'_2 = \pi'_L(K_2 \cap P'_2),$$
$$\pi'_L(-K_1) \cap (-Q'_1) = \pi'_L((-K_1) \cap (-Q'_1))$$

are disjoint for all $\gamma \in]0, \varepsilon_3[$. Hence $\pi'_L(K_2) \ne \pi'_L(-K_1)$ for all $\gamma \in]0, \varepsilon_3[$.

Finally, with $\varepsilon_0 = \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$, we have

$$\pi'_L(K_1) \ne \pi'_L(K_2) \ne \pi'_L(-K_1) \quad \text{for all} \quad \gamma \in]0, \varepsilon_0[,$$

in contradiction with the condition that $\pi'_L(K_2)$ equals one of the sets $\pi'_L(K_1)$, $\pi'_L(-K_1)$. Thus $[x_2, z_2]$ coincides with $[x_1, z_1]$ or with $[-x_1, -z_1]$.

5. Our concluding statement (in continuation of **4**) is that $K_2 = K_1$ or $K_2 = -K_1$.

Indeed, assume for a moment that $K_1 \ne K_2 \ne -K_1$. Since $K_1 \ne K_2$, Theorem 2 implies that K_1 has an exposed diameter $[u_1, v_1]$ that is not an exposed diameter of K_2. Then **4** above implies that $[-v_1, -u_1]$ is a common exposed diameter of K_2 and $-K_1$. In particular, $[u_1, v_1] \ne [-v_1, -u_1]$. Similarly, $K_2 \ne -K_1$ implies the existence of an exposed diameter $[-v_0, -u_0]$ of $-K_1$ which is not an exposed diameter of K_2, while $[u_0, v_0]$ is a common exposed diameter of K_1 and K_2. Again, $[u_0, v_0] \ne [-v_0, -u_0]$. By Lemma 1, $[u_0, v_0]$ and $[u_1, v_1]$ are not parallel.

Denote by w the middle point of $[u_0, v_0]$ and consider the sets $K'_1 = K_1 - w$ and $K'_2 = K_2 - w$. We observe that $w \ne o$ because of $[u_0, v_0] \ne [-v_0, -u_0]$. The origin o is the middle point of the exposed diameter $[u_0 - w, v_0 - w]$ of K'_1, which is also an exposed diameter of K'_2. By **4** above (with $[u_0, v_0]$ instead of $[x_0, z_0]$), we see that every exposed diameter of K'_2 is an exposed diameter of K'_1 or $-K'_1$. In particular, the exposed diameter $[-v_1 - w, -u_1 - w]$ of K'_2 should coincide either with the exposed diameter $[u_1 - w, v_1 - w]$ of K'_1 or with the exposed diameter $[-v_1 + w, -u_1 + w]$ of $-K'_1$.

On the other hand,

$$[-v_1 - w, -u_1 - w] \ne [u_1 - w, v_1 - w]$$

due to $[-v_1, -u_1] \neq [u_1, v_1]$, and

$$[-v_1 - w, -u_1 - w] \neq [-v_1 + w, -u_1 + w]$$

because of $w \neq o$. The obtained contradiction shows that $K_2 = K_1$ or $K_2 = -K_1$, which concludes the proof of Case I for $m = n - 1$.

B) Now we assume that $2 \le m < n - 1$. Let $M \subset \mathbb{R}^n$ be a plane of dimension $m + 1$. For any plane $L \subset M$ of dimension m, we can express π_L as the composition $\pi_L = \pi' \circ \pi_M$, where π' is the orthogonal projection of M onto L. This observation and condition 2) of the theorem imply that the orthogonal projections of the sets $\pi_M(K_1)$ and $\pi_M(K_2)$ on every m-dimensional plane $L \subset M$ are homothetic. By the proved above (with $m + 1$ instead of n), the sets $\pi_M(K_1)$ and $\pi_M(K_2)$ are homothetic. Since this argument holds for every $(m+1)$-dimensional plane in $\mathbb{R}^n$, we can replace m with $m + 1$ in condition 2) of the theorem. Repeating this argument finitely many times, we see that the orthogonal projections of K_1 and K_2 on each hyperplane of $\mathbb{R}^n$ are homothetic. By the proved above, K_1 and K_2 are homothetic themselves.

Case II. At least one of the sets K_1 and K_2 is unbounded and $3 \le m \le n - 1$.

Let, for example, K_1 be unbounded. Then $\operatorname{rec} K_1 \neq \{o\}$. Choose a closed halfline h with apex o that lies in $\operatorname{rec} K_1$ and an m-dimensional subspace L that contains h. Then $h \subset \operatorname{rec} \pi_L(K_1)$, which shows that $\pi_L(K_1)$ is unbounded. Since $\pi_L(K_2)$ is homothetic to $\pi_L(K_1)$, the set $\pi(K_2)$ is also unbounded, which implies that K_2 is unbounded.

6. We state that $\operatorname{lin} K_1 = \operatorname{lin} K_2$.

Indeed, assume, for example, that $\operatorname{lin} K_1$ contains a line l through o that does not belong to $\operatorname{lin} K_2$. Then l does not lie entirely in $\operatorname{rec} K_2$, since otherwise l would belong to $\operatorname{lin} K_2$. Let h be a halfline of l with apex o that does not lie in $\operatorname{rec} K_2$. Because $\operatorname{rec} K_2$ is a closed convex cone with apex o, there is a closed halfspace Q that contains $\operatorname{rec} K_2$ and is disjoint from $h \setminus \{o\}$. Clearly, $o \in \operatorname{bd} Q$. Choose an $(n - m)$-dimensional subspace N in $\operatorname{bd} Q$, and denote by L the orthogonal complement to N. Clearly, the line $\pi_L(l)$ lies in $\operatorname{rec} \pi_L(K_1)$ and does not lie in $\operatorname{rec} \pi_L(K_2)$, which belongs to $L \cap Q$. The last is impossible because $\pi_L(K_1)$ and $\pi_L(K_2)$ are homothetic by condition 2). Hence $\operatorname{lin} K_1 \subset \operatorname{lin} K_2$. Similarly, $\operatorname{lin} K_2 \subset \operatorname{lin} K_1$.

7. Due to **6** above, both K_1 and K_2 can be expressed as

$$K_1 = \operatorname{lin} K_1 \oplus (K_1 \cap M), \quad K_2 = \operatorname{lin} K_1 \oplus (K_2 \cap M), \tag{6}$$

where the subspace M is the orthogonal complement of $\operatorname{lin} K_1$ and both sets $K_1 \cap M$ and $K_2 \cap M$ are line-free.

First assume that $\dim M \le m$. In this case, we choose an m-dimensional subspace $L \subset \mathbb{R}^n$ that contains M. Clearly,

$$\pi_L(K_i) = (\operatorname{lin} K_1 \cap L) \oplus (K_i \cap M), \quad i = 1, 2.$$

Then $K_1 \cap M$ and $K_2 \cap M$ are homothetic because the sets $\pi_L(K_1)$ and $\pi_L(K_2)$ are homothetic by the hypothesis. This and (6) imply that K_1 and K_2 are homothetic themselves.

Now assume that $\dim M > m$. Since $K_1 \cap M$ is line-free, it contains an exposed point x. Translating K_1 on the vector $-x$, we may assume that o is an exposed point of K_1. Let N be a subspace of M of dimension $\dim M - 1$ that supports $K_1 \cap M$ such that $N \cap (K_1 \cap M) = \{o\}$. Denote by N_+ and N_- the opposite closed halfplanes of M bounded by N. Let, for example, $K_1 \cap M \subset N_+$. Denote by l the 1-dimensional subspace of M orthogonal to N. Choose an m-dimensional subspace S of M that contains l. By the above, $\pi_S(K_1 \cap M) \subset S \cap N_+$.

7a. If $\pi_S(K_2 \cap M)$ is positively homothetic to $\pi_S(K_1 \cap M)$, then the recession cones of $\pi_S(K_1 \cap M)$ and $\pi_S(K_2 \cap M)$ coincide. This shows that for any other m-dimensional subspace S' of M that contains l, the orthogonal projections of $K_1 \cap M$ and $K_2 \cap M$ on S' are positively homothetic. Since $m \geq 3$, it follows from [13] that $K_1 \cap M$ and $K_2 \cap M$ are positively homothetic, and (6) implies that K_1 and K_2 are positively homothetic themselves.

7b. If $\pi_S(K_2 \cap M)$ is negatively homothetic to $\pi_S(K_1 \cap M)$, then the recession cones of $\pi_S(K_1 \cap M)$ and $\pi_S(K_2 \cap M)$ are symmetric about o. This shows that for any other m-dimensional subspaces S' of M that contains l, the orthogonal projections of $K_1 \cap M$ and $K_2 \cap M$ on S' are negatively homothetic. Since $m \geq 3$, it follows from [13] that $K_1 \cap M$ and $K_2 \cap M$ are negatively homothetic, and (6) implies that K_1 and K_2 are negatively homothetic themselves. $\square$

4 Proof of Corollary 1

Because 1) obviously implies 2), it remains to show that 2) $\Rightarrow$ 1). Let compact convex sets K_1 and K_2 in $\mathbb{R}^n$ satisfy condition 2) of the corollary. Choose any 2-dimensional subspace $L \subset \mathbb{R}^n$. Since $\dim(L + S) \leq r + 2 \leq m$, there is an m-dimensional subspace M that contains $L + S$. By condition 2), $\pi_M(K_1)$ and $\pi_M(K_2)$ are homothetic. This implies that the orthogonal projections of the sets $\pi_M(K_1)$ and $\pi_M(K_2)$ onto L are homothetic. Because $\pi_L = \pi' \circ \pi_M$, where π' is the orthogonal projection of M onto L, we conclude that $\pi_L(K_1)$ and $\pi_L(K_2)$ are homothetic. Now Theorem 1 (with $m = 2$) implies that K_1 and K_2 are homothetic themselves.

If K_1 and K_2 are closed convex sets that satisfy condition 2) of the corollary, then repeating the argument above, with any 3-dimensional subspace $L \subset \mathbb{R}^n$ and the respective inequality $\dim(L+S) \leq r+3 \leq m$, we obtain the homothety of K_1 and K_2. $\square$

References

[1] R. J. Gardner, A. Volčič, *Convex bodies with similar projections,* Proc. Amer. Math. Soc. **121** (1994), 563–568.

[2] V. P. Golubyatnikov, *On the unique determination of visible bodies from their projections,* (Russian) Sibirsk. Mat. Zh. **29** (1988) 92–96. English translation: Siberian Math. J. **29** (1988), 761–764.

[3] V. P. Golubyatnikov, *On uniqueness of reconstruction of the form of convex and visible bodies from their projections,* (Russian) Algebra and Analysis (Irkutsk, 1989). English translation: Amer. Math. Soc. Transl. Ser. 2, Vol. 163, pp. 35–45, Amer. Math. Soc., Providence, 1995.

[4] H. Hadwiger, *Seitenrisse konvexer Körper und Homothetie,* Elem. Math. **18** (1963), 97–98.

[5] V. L. Klee, Extremal structure of convex sets, Arch. Math. 8 (1957), 234–240.

[6] H. Martini, V. Soltan, *Antipodality properties of finite sets in Euclidean space,* Discrete Math. **290** (2005), 221–228.

[7] S. Nakajima, *Über homothetische Eiflächen,* Jap. J. Math. **7** (1930), 167–169.

[8] M. H. Nguyên, V. Soltan, *Lower bounds for the numbers of antipodal pairs and strictly antipodal pairs of vertices in a convex polytope,* Discrete Comput. Geom. **11** (1994), 149–162.

[9] C. M. Petty, J. R. McKinney, *Convex bodies with circumscribing boxes of constant volume,* Portugal. Math. **44** (1987), 447–455.

[10] C. A. Rogers, *Sections and projections of convex bodies,* Portugal. Math. **24** (1965), 99–103.

[11] V. Soltan, M. H. Nguyên, *Lower bounds for the number of extremal and exposed diameters of a convex body,* Studia Sci. Math. Hungar. **28** (1993), 99–104.

[12] V. Soltan, *Affine diameters of convex bodies–a survey,* Expo. Math. **23** (2005), 47–63.

[13] V. Soltan, *Sections and projections of homothetic convex bodies,* J. Geom. **84** (2005), 152–163.

[14] V. Soltan, *Addition and subtraction of homothety classes of convex sets,* Beiträge Algebra Geom. **47** (2006), 351–361.

[15] W. Süss, *Zu Minkowskis Theorie von Volumen und Oberfläche,* Math. Ann. **101** (1929), 253–260.

[16] W. Süss, *Zusammensetzung von Eikörpern und Homothetische Eiflächen,* Tôhoku Math. J. **35** (1932), 47–50.

[17] S. Straszewicz, *Über exponierte Punkte abgeschlossener Punktmengen,* Fund. Math. **24** (1935), 139–143.

[18] R. J. Webster, *Convexity,* Oxford University Press, Oxford, 1994.